

VIRTUAL COVERS OF LINKS II

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ABSTRACT. A fibered concordance of knots, introduced by Harer, is a concordance between fibered knots that is well-behaved with respect to the fibrations. We consider semi-fibered concordance of two component ordered links $L = J \sqcup K$ with J fibered. These are concordances that restrict to fibered concordances on the first component. Motivated by some examples of Gompf-Scharlemann-Thompson, we further limit our attention to those links L where K is “close to” a fiber of J . Such L are studied with virtual covers, where a virtual knot v is associated to L . We show that the concordance class of v is a semi-fibered concordance invariant. This gives obstructions for certain slice and ribbon discs for the K component. Further applications are to injectivity of satellite operators in semi-fibered concordance and to knots in fibered 3-manifolds.

In [14], Gompf-Scharlemann-Thompson described an infinite family of two component slice links $L_n = J_0 \sqcup V_n$ that are unknown to be ribbon. The knot J_0 is a square knot and hence is fibered. Since L_n is concordant to the two component unlink and the unknot is fibered, it is natural to investigate concordances between J_0 and the unknot that behave nicely with respect to the fibrations. Such a notion of *fibered concordance* of knots was introduced by Harer [17]. Thus we consider *semi-fibered concordance* of two component links with first component fibered: concordances restricting to a fibered concordance of the first component.

Scharlemann [26] later showed how to arrange V_n as a simple closed curve on a fiber of J_0 . Such links are examples of links in *special Seifert form* (SSF) introduced in [6]. These are two component links $L = J \sqcup K$ with J fibered and K lying “close to” a fiber (see Section 1.4). Such links may be studied with *virtual covers*, where a virtual knot v is associated to L so that it functions essentially as an invariant of L . In [4] it was shown that v can detect geometric properties of L , such as if L is non-split or non-invertible.

These observations provide the setting for this sequel to [4]. The main theorem is that the concordance class of v as a virtual knot is a semi-fibered concordance invariant of two component links L in SSF (see Section 2.1). An obstruction to the existence of certain kinds of slice and ribbon discs is obtained (see Section 3). We give an example where it is more discriminating than Cochran’s β invariant [7]. The obstruction vanishes for L_n from [14].

Our second application is to the injectivity of satellite operators. Section 4 gives a combinatorial condition on v under which the untwisted satellite operator $J \sqcup K \rightarrow J \sqcup P(K)$ acts injectively in semi-fibered concordance, where P is a pattern of non-zero winding number and $J \sqcup K$ is in SSF. Virtual covers can also be applied to knots in closed fibered 3-manifolds.

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Applications are sketched in Section 5. Section 1 contains a review of virtual covers, virtual knot concordance, and concordance invariants of virtual knots.

1. BACKGROUND

1.1. Links and Virtual Knots. We will assume the reader has some familiarity with virtual knot theory. Recall that there are four models for virtual knots: (1) virtual knot diagrams modulo extended Reidemeister moves [21], (2) Gauss diagrams modulo diagrammatic versions of the Reidemeister moves [16], (3) abstract link diagrams modulo Kamada-Kamada equivalence [20], (4) knots in thickened surfaces modulo stabilization/destabilization and stable diffeomorphism [23]. Given a representative R from any of these four models of virtual knots, we will denote by $\kappa(R)$ the equivalence class of virtual knot diagrams from model (1). Equivalence of (ordered, oriented) links in \mathbb{S}^3 and oriented virtual knots is denoted by “ \equiv ”.

1.2. Concordance of Knots and Virtual Knots. We work throughout in the smooth category. Two oriented knots K_0 and K_1 in \mathbb{S}^3 are said to be concordant in \mathbb{S}^3 if there is an embedded oriented annulus A in $\mathbb{S}^3 \times \mathbb{I}$ such that $A \cap \mathbb{S}^3 \times \{i\} = (-1)^i K_i$, where $-K$ denotes a change of orientation. If K_0 and K_1 are concordant, we write $K_0 \simeq K_1$. A concordance can be realized combinatorially as a sequence of births b (local minimums), deaths d (local maximums), and saddle moves s such that $\#b - \#s + \#d = 0$.

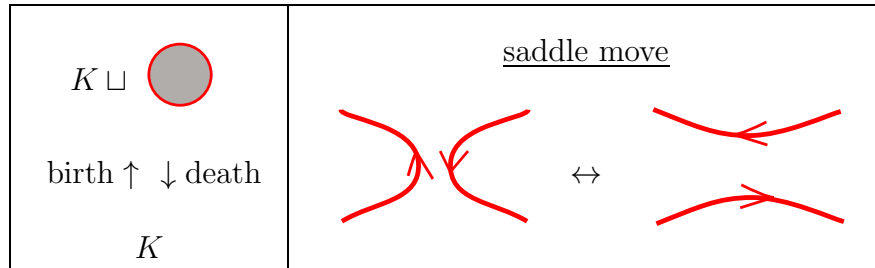


FIGURE 1. Births, Deaths, and Saddles.

Let Σ be a compact connected oriented (c.c.o.) surface. For knots in $\Sigma \times \mathbb{R}$, we have notion of concordance due to Turaev [30]. We denote a knot K in a 3-manifold N by K^N . For $i = 0, 1$, let \mathfrak{k}_i be knots in $\Sigma_i \times \mathbb{R}$. Then $\mathfrak{k}_0^{\Sigma_0 \times \mathbb{R}}, \mathfrak{k}_1^{\Sigma_1 \times \mathbb{R}}$ are *concordant* if there is a c.c.o. 3-manifold M , an embedding of the surface $\Sigma_0 \sqcup -\Sigma_1 \rightarrow \partial M$, and a properly embedded oriented annulus $\mathfrak{a} : \mathbb{S}^1 \times \mathbb{I} \rightarrow M \times \mathbb{R}$ such that for $i = 0, 1$, $\mathfrak{a} \cap (\Sigma_i \times \mathbb{R}) = (-1)^i \mathfrak{k}_i^{\Sigma_i \times \mathbb{R}}$. Again we denote concordant knots in thickened surfaces by $\mathfrak{k}_0^{\Sigma_0 \times \mathbb{R}} \simeq \mathfrak{k}_1^{\Sigma_1 \times \mathbb{R}}$.

Two oriented virtual knots v_0, v_1 are *concordant* if they are obtained from one another by a finite sequence of extended Reidemeister moves, births b , deaths d , and saddle moves s satisfying $\#b - \#s + \#d = 0$. This combinatorial definition, introduced in [10], was shown to be equivalent to a geometric formulation in [3]: the concordance relation for virtual knots is equivalent to concordance relation of knots in thickened surfaces together with stabilization/destabilization (see [3], Lemma 4.6).

1.3. Concordance Invariants of Virtual Knots. We will use two invariants to separate concordance classes of virtual knots: the Henrich-Turaev (HT) polynomial and the slice genus. Let v be an oriented virtual knot diagram. Let G be a Gauss diagram for v and x an arrow of G . Let $\bowtie(G)$ be the set of arrows of G . Suppose that $a \in \bowtie(G) \setminus \{x\}$ intersects x . Define $\text{int}_x(a) = \pm 1$ according to Figure 2. Then define:

$$\text{index}(x) = \sum_a \text{sign}(a) \text{int}_x(a), \quad w_v(t) = \sum_{\substack{x \in \bowtie(G) \\ \text{index}(x) \neq 0}} \text{sign}(x) t^{|\text{index}(x)|},$$

where $\text{sign}(a) = \pm 1$ denotes the local writhe of the crossing of a . The polynomial $w_v(t)$ is called the HT^1 polynomial [18, 30].

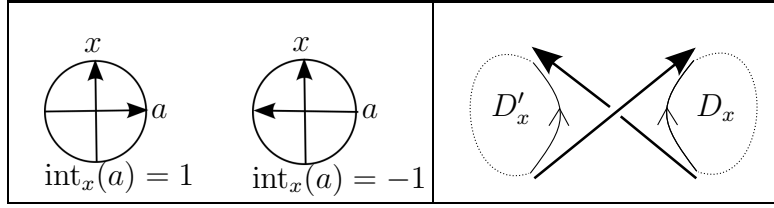


FIGURE 2. (Left) Definition of $\text{int}_x(a)$ and (Right) distinguished halves.

Theorem 1. *The HT polynomial is a concordance invariant of virtual knots.*

Proof. Suppose that $\mathfrak{k}_0^{\Sigma_0 \times \mathbb{R}} \asymp \mathfrak{k}_1^{\Sigma_1 \times \mathbb{R}}$, where each \mathfrak{k}_i has a regular projection to Σ_i . Then we may determine $\kappa(\mathfrak{k}_i)$ by finding a Gauss diagram G_i for \mathfrak{k}_i on Σ_i in the usual way. If $x \in \bowtie(G_i)$, $|\text{index}(x)|$ may be computed by performing the oriented smoothing at x (see Figure 2) and taking the algebraic intersection number of the resulting two curves D_x, D'_x on Σ (for details, see [5]). If D is the immersed curve of the diagram of \mathfrak{k}_i in Σ_i , then $[D] = [D_x] + [D'_x]$ in $H_1(\Sigma; \mathbb{Z})$. Hence $|D \cdot D_x| = |D_x \cdot D'_x|$. This implies that:

$$w_{\kappa(\mathfrak{k}_i)}(t) = \sum_{x \in \bowtie(G_i), D \cdot D_x \neq 0} \text{sign}(x) t^{|D_x \cdot D|}.$$

In [30], Turaev defines two concordance invariants u_+, u_- of knots in thickened surfaces. The above formula shows that $w_{\kappa(\mathfrak{k}_i)}(t) = u_+(\mathfrak{k}_i)(t) + u_-(\mathfrak{k}_i)(t)$. Hence, $w_{\kappa(\mathfrak{k}_0)}(t) = w_{\kappa(\mathfrak{k}_1)}(t)$. \square

If the condition that $\#b - \#s + \#d = 0$ is removed from virtual knot concordance, then we have *virtual knot cobordism*. The *slice genus* of a virtual knot v is the smallest genus of all formal cobordisms taking v to the unknot. For a virtual knot with all positive crossings, i.e. a positive virtual knot, the slice genus can be computed using a generalization of Rasmussen's theorem [10]. Recall that if the oriented smoothing is preformed at all classical crossings of v , then the resulting set of immersed curves is in one-to-one correspondence with the *virtual Seifert circles* of v . The virtual Seifert circles of a classical knot are the Seifert circles.

Theorem 2 (Generalization of Rasmussen [10]). *Let v be a virtual knot diagram whose classical crossings are all positively signed. Let r be the number of virtual Seifert circles of v and n the number of classical crossings of v . Then the slice genus is $(-r + n + 1)/2$.*

¹This definition of the HT polynomial is slightly different from the definition in [18, 30]

1.4. Virtual Covers of Links. Here we review virtual covers [4]. Let N be a c.c.o. 3-manifold admitting a regular orientation preserving (o.p.) covering projection $\Pi : \Sigma \times \mathbb{R} \rightarrow N$, where Σ is a c.c.o. 2-manifold. A *lift by* Π of K^N is an oriented knot $\mathfrak{k}^{\Sigma \times \mathbb{R}}$ such that $\Pi(\mathfrak{k}^{\Sigma \times \mathbb{R}}) = K^N$, with orientations preserved. The triple $(\mathfrak{k}^{\Sigma \times \mathbb{R}}, \Pi, K^N)$ specifying a lift is called a *virtual cover*. The virtual knot $\kappa(\mathfrak{k}^{\Sigma \times \mathbb{R}})$ is called the *associated virtual knot*. Let $\Upsilon(K^N)$ denote the set of all associated virtual knots for all lifts by Π of K^N . If $|\Upsilon(K^N)| = 1$, then we call the unique element the *invariant associated virtual knot*. The following lemma from [4], is essentially the main result in [6] in a more compact form.

Lemma 3. *Suppose there are virtual covers $(\mathfrak{k}_0^{\Sigma \times \mathbb{R}}, \Pi, K_0^N), (\mathfrak{k}_1^{\Sigma \times \mathbb{R}}, \Pi, K_1^N)$ with invariant associated virtual knots v_0, v_1 , respectively. If $K_0^N \rightleftharpoons K_1^N$, then $v_0 \rightleftharpoons v_1$.*

The canonical example of a virtual cover is a knot K in the complement of a fibered knot J where $\text{lk}(J, K) = 0$. Set $N_J = \overline{\mathbb{S}^3 \setminus V(J)}$, where $V(*)$ denotes a tubular neighborhood of $*$. Since J is fibered, it has a Seifert surface Σ_J such that the pair $(\overline{\mathbb{S}^3 \setminus V(\Sigma_J)}, \overline{\mathbb{S}^3 \setminus V(\Sigma_J)} \cap \partial N_J)$ is diffeomorphic as a pair to $(\Sigma_J \cap N_J, \partial(\Sigma_J \cap N_J)) \times \mathbb{I}$ [22]. Then N_J may be identified with a mapping torus $(\Sigma_J \cap N_J) \times \mathbb{I} / \psi$ where $\psi : (\Sigma_J \cap N_J) \rightarrow (\Sigma_J \cap N_J)$ is an o.p. diffeomorphism. This gives a regular covering projection $\Pi_J : (\Sigma_J \cap N_J) \times \mathbb{R} \rightarrow N_J$.

Since $\text{lk}(J, K) = 0$, there is a virtual cover $(\mathfrak{k}^{\Sigma_J \times \mathbb{R}}, \Pi_J, K^{N_J})$. The associated virtual knot is invariant when K is in *special Seifert form* (SSF) with respect to Σ_J . Briefly, we say K is in SSF with respect to Σ_J if its image can be decomposed as a disjoint union of a finite number of embedded intervals on Σ_J and a finite number of “crossings”. Each “crossing” consists of a pair of disjoint arcs lying in different hemispheres of an embedded 3-ball B satisfying $B \cap \Sigma_J \approx \mathbb{B}^2$. A formal definition is given in [6]. See also [4] for further discussion.

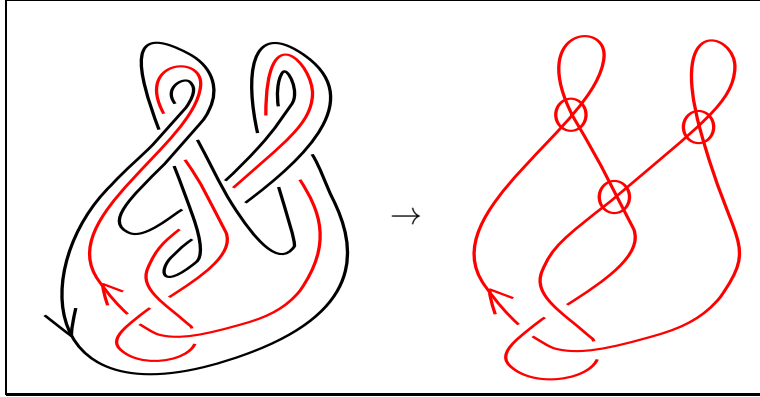


FIGURE 3. A link $J \sqcup K$ in SSF (left) and its invariant associated virtual v (right).

If K is in SSF with respect to a fiber Σ_J , let $[K; \Sigma_J]$ denote the oriented knot diagram on Σ_J made of the arcs and crossings of the SSF. In [6] it was shown that $v \rightleftharpoons \kappa([K; \Sigma_J])$ is the invariant associated virtual knot. See Figure 3 for a typical example. The main result of [4] showed that v functions essentially as an invariant of links in SSF:

Theorem 4. *For $i = 0, 1$, let $L_i = J_i \sqcup K_i$ be oriented ordered links with J_i a fibered link, K_i in SSF with respect to a fiber Σ_i of J_i . For $i = 0, 1$, let v_i be the invariant associated virtual knot for L_i . If $L_0 \rightleftharpoons L_1$, then $v_0 \rightleftharpoons v_1$.*

2. SEMI-FIBERED CONCORDANCE

This section gives the precise definitions of fibered concordance and semi-fibered concordance. It is shown that the concordance class of the invariant associated virtual knot is a semi-fibered concordance invariant of two component links in SSF. Furthermore, it is shown this concordance class can be used to separate an infinite number of semi-fibered concordance classes of two component links.

2.1. Semi-Fibered Concordance of Links. Let J_0, J_1 be oriented fibered knots in \mathbb{S}^3 with fiber bundle projections $p_0 : \overline{\mathbb{S}^3 \setminus V(J_0)} \rightarrow \mathbb{S}^1$, $p_1 : \overline{\mathbb{S}^3 \setminus V(J_1)} \rightarrow \mathbb{S}^1$, resp. Suppose there is an annulus A embedded in $\mathbb{S}^3 \times \mathbb{I}$ such that $A \cap \mathbb{S}^3 \times \{i\} = (-1)^i J_i$ for $i = 0, 1$ and such that there is a fiber bundle projection $p : \overline{\mathbb{S}^3 \times \mathbb{I} \setminus V(A)} \rightarrow \mathbb{S}^1$ where $p|_{\mathbb{S}^3 \times \{i\}} = p_i$ for $i = 0, 1$. Then J_0 and J_1 are said to be *fibered concordant*, denoted $J_0 \asymp_f J_1$ [17].

When $J_0 \asymp_f J_1$, there is a covering projection $\Pi : H \times \mathbb{R} \rightarrow \overline{\mathbb{S}^3 \times \mathbb{I} \setminus V(A)}$, where the fiber H is a c.c.o. 3-manifold. Moreover, there are oriented Seifert surfaces Σ_0 of J_0 and Σ_1 of J_1 such that $\Sigma_0 \sqcup -\Sigma_1 \hookrightarrow \partial H$ and $\Pi|_{\Sigma_0 \times \mathbb{R}}, \Pi|_{\Sigma_1 \times \mathbb{R}}$ are the infinite cyclic covers of J_0, J_1 , respectively.

Remark 2.1. It is known that there are knots that are not concordant to a fibered knot [25]. Every fibered knot in \mathbb{S}^3 is fiber concordant to a hyperbolic fibered knot [29].

Let $L_0 = J_0 \sqcup K_0$, $L_1 = J_1 \sqcup K_1$ be two component links with J_i fibered and $J_0 \asymp_f J_1$ via an annulus A_J in $\mathbb{S}^3 \times \mathbb{I}$. If $K_0 \asymp K_1$ via an annulus A_K in $\mathbb{S}^3 \times \mathbb{I}$ with $A_K \cap A_J = \emptyset$, then we will say that L_0 and L_1 are *semi-fibered concordant*. This will be denoted by $L_0 \asymp_{sf} L_1$.

Lemma 5. *Let $L = J \sqcup K$ with J fibered and K in SSF with respect to some fiber Σ_J and v the invariant associated virtual knot. If $L \hookrightarrow J_1 \sqcup K_1$, then $L \asymp_{sf} J_1 \sqcup K_1$, K_1 is in SSF with respect to some fiber Σ_{J_1} , and $v \hookrightarrow \kappa([K_1; \Sigma_{J_1}])$.*

Proof. Let $F : \mathbb{S}^3 \times \mathbb{I} \rightarrow \mathbb{S}^3$ be the ambient isotopy taking L to $J_1 \sqcup K_1$. Let A_J be the annulus defined as the image of the map $a_J : \mathbb{S}^1 \times \mathbb{I} \rightarrow \mathbb{S}^3 \times \mathbb{I}$, $a_J(z, t) = (F(J(z), t), t)$. Similarly define an embedded annulus A_K . Then $J \asymp_f J_1$ via A_J and $A_K \cap A_J = \emptyset$. Thus $L \asymp_{sf} J_1 \sqcup K_1$. Set $\Sigma_{J_1} = F(\Sigma_J, 1)$. Then $K_1 = F(K_0, 1)$ is in SSF with respect to Σ_{J_1} (see [4], Lemma 2). The last claim follows from Theorem 4. \square

Theorem 6 (Main Theorem). *Let $L_0 = J_0 \sqcup K_0$, $L_1 = J_1 \sqcup K_1$ be two component links with J_0, J_1 fibered. Suppose for $i = 0, 1$, K_i in SSF with respect to some fiber Γ_i of J_i and let v_i be the invariant associated virtual knot v_i . If $L_0 \asymp_{sf} L_1$, then $v_0 \asymp v_1$.*

Proof. Let $A_J, A_K, N, \Pi, H, \Sigma_0, \Sigma_1$ be as in the definitions above. The fibers Γ_i and Σ_i are minimal genus Seifert surfaces. Minimal genus Seifert surfaces of fibered knots J_i are unique up to ambient isotopy in \mathbb{S}^3 acting as the identity on J_i [31]. Thus for $i = 0, 1$, we may use Lemma 5 to move Γ_i and K_i so that we have a new knot in SSF with respect to Σ_i . As this does not affect the semi-fibered concordance classes of L_i or the concordance class of v_i , we may as well assume from the beginning that $\Gamma_i = \Sigma_i$.

Now, there are virtual covers $(\mathfrak{k}_i^{\Sigma_i \times \mathbb{R}}, \Pi|_{\Sigma_i \times \mathbb{R}}, K_i^{N_{J_i}})$, $i = 0, 1$. By the homotopy lifting theorem, A_K lifts to a smoothly embedded annulus $\mathfrak{a} : \mathbb{S}^1 \times \mathbb{I} \rightarrow H \times \mathbb{R}$ such that $\mathfrak{a}(\mathbb{S}^1, 0)$ is identified with $\mathfrak{k}_0^{\Sigma_0 \times \mathbb{R}}$. Set $\mathfrak{l} = \mathfrak{a}(\mathbb{S}^1, 1)$ so that we have another virtual cover $(\mathfrak{l}^{\Sigma_1 \times \mathbb{R}}, \Pi|_{\Sigma_1 \times \mathbb{R}}, K_1^{N_{J_1}})$.

Orient \mathfrak{l} appropriately, so that it matches the orientation of lifts by $\Pi|_{\Sigma_1 \times \mathbb{R}}$ of $K_1^{N_{J_1}}$.

Then \mathfrak{k}_0 and \mathfrak{l} are concordant as knots in thickened surfaces. Hence we have that $\kappa(\mathfrak{k}_0) \asymp \kappa(\mathfrak{l})$ as virtual knots (by [3], Lemma 4.6). Since equivalent virtual knots are concordant, and all lifts of K_1 by $\Pi|_{\Sigma_1 \times \mathbb{R}}$ stabilize to the same virtual knot (by hypothesis), $v_1 \asymp \kappa(\mathfrak{l})$ and $v_0 \asymp \kappa(\mathfrak{k}_0)$. This completes the proof. \square

The following theorem shows that the associated virtual knots distinguish a large set of semi-fibered concordance classes. Moreover, it shows that the HT polynomial is useful at separating these classes. It is inspired by [19], Theorem 1, and [30], Theorem 1.6.1.

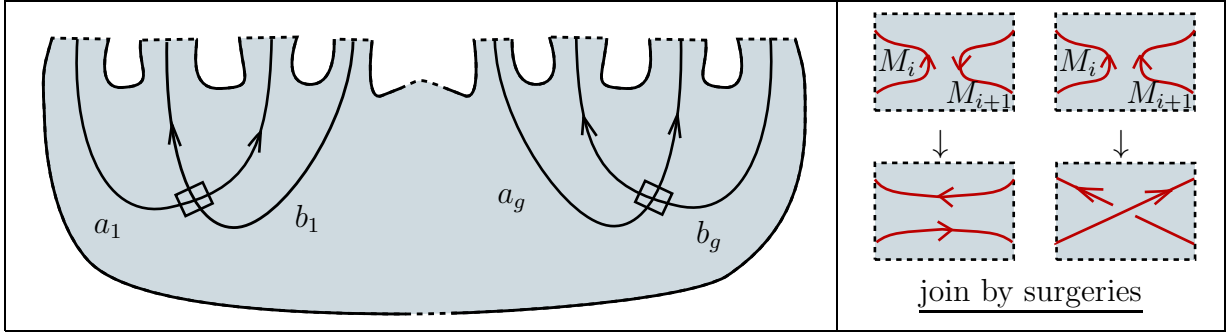


FIGURE 4. A fiber Σ_J in disc-band form.

Theorem 7. *Let J be a fibered knot and Σ_J a fiber of genus g in disc-band form, with symplectic basis $a_1, b_1, \dots, a_g, b_g$. Let $h = \sum_{i=1}^g p_i a_i + q_i b_i \in H_1(\Sigma_J; \mathbb{Z})$ such that for all i , $p_i, q_i \neq 0$, and $\gcd(|p_i|, |q_i|) = 1$. Then there is an infinite set of pairwise non-semi-fibered concordant non-split links $J \sqcup K_k$ with K_k in SSF with respect to Σ_J , $[K_k; \Sigma_J]$ represents h , and $w_{v_k}(t) \neq 0$, where v_k is the invariant associated virtual knot.*

Proof. Since Σ_J is in disc-band form [2], we may assume that the Σ_J and the symplectic basis is as depicted in Figure 4. Curves representing a_i, b_i are also denoted as a_i, b_i . For each i , draw p_i (resp. q_i) simple closed curves on Σ_J parallel to a_i (resp. b_i). This makes $|p_i q_i|$ intersections in a neighborhood of the point $a_i \cap b_i$ (see Figure 5, left). For each intersection, substitute the same picture in a purple box from Figure 5, center: top left if $p_i, q_i > 0$, top right if $p_i, q_i < 0$, bottom left if $p_i > 0$ and $q_i < 0$, bottom right if $p_i < 0$ and $q_i > 0$.

Since $\gcd(|p_k|, |q_k|) = 1$, we have g knot diagrams $[M_1; \Sigma_J], \dots, [M_g; \Sigma_J]$, oriented by choice of a purple box. For $k \geq 1$, insert the long $(2k+1, 2)$ torus knot in a small arc on $[M_1; \Sigma_J]$ away from any of the purple boxes (as in the orange circle in Figure 5). These knot diagrams may be surgered together from left to right in Σ_J as in Figure 4, right. Call the resulting knot diagram $[K_k; \Sigma_J]$. Clearly, this corresponds to an oriented knot K_k in SSF with respect to Σ_J . Now compute $w_{v_k}(t) = 2t \cdot \sum_{i=1}^g |p_i q_i| \neq 0$. Hence, $J \sqcup K_k$ is non-split [6]. Lastly, Theorem 2 implies that increasing k by 1 increases the slice genus of v_k by 1. \square

3. RIBBON AND SLICE OBSTRUCTIONS

Now we will use the results of the previous section to identify slice and ribbon obstructions. Recall that a knot in \mathbb{S}^3 is said to be *slice* if $K \asymp \bigcirc$, with \bigcirc bounding an embedded disc.

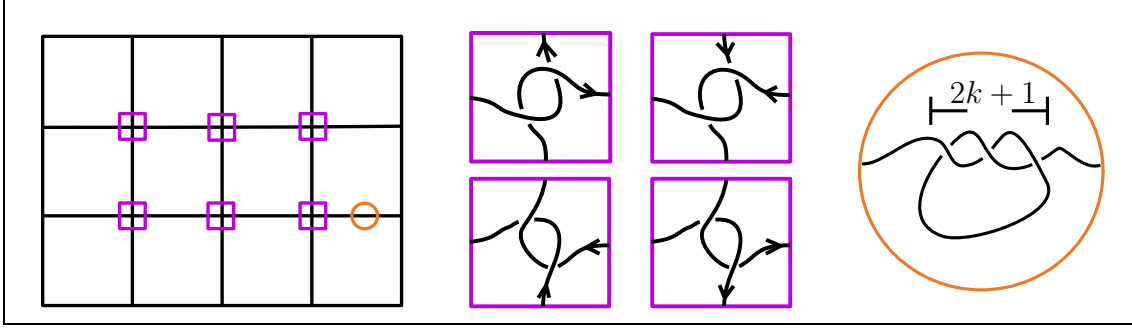


FIGURE 5. The construction used in the proof of Theorem 7.

A knot is said to be *ribbon* in \mathbb{S}^3 if it bounds an immersed disc \mathbb{B}^2 in \mathbb{S}^3 having only ribbon singularities: the pre-image of any singular arc is two disjoint simple arcs a_b and a_i , where a_b intersects $\partial\mathbb{B}^2$ in its endpoints and $a_i \subseteq \text{int}(\mathbb{B}^2)$. We will use the same definition for slice and ribbon in any oriented 3-manifold N : just replace \mathbb{S}^3 with N .

A ribbon knot K^N is a band connected sum in N of the boundaries of n disjoint discs embedded in N (i.e. an n component unlink in N). Performing saddle moves on the bands gives n disjoint 2-discs which may be eliminated with deaths. Likewise, a virtual knot v is said to be *ribbon* if there is concordance of v with the unknot consisting of a sequence of extended Reidemeister moves, deaths, and saddle moves. Thus, if $\mathfrak{k}^{\Sigma \times \mathbb{R}}$ is a ribbon knot in the thickened c.c.o. surface $\Sigma \times \mathbb{R}$, then its stabilization $\kappa(\mathfrak{k}^{\Sigma \times \mathbb{R}})$ is a ribbon virtual knot.

Theorem 8. (with R. Todd) Let $L = J \sqcup K$ with J fibered and K in SSF with respect to some fiber Σ of J and v the invariant associated virtual knot.

- (1) If $L \asymp_{sf} J_0 \sqcup \bigcirc$, with \bigcirc bounding an embedded disc in the complement of J_0 , then $v \asymp \bigcirc$ (i.e. it is a slice virtual knot).
- (2) If K bounds a ribbon disc disjoint from J , then $L \asymp_{sf} J \sqcup \bigcirc$, where \bigcirc bounds a disc in the complement of J and v is a ribbon virtual knot.

Proof. The first claim is an immediate consequence of Theorem 6. For the second claim, let A_J be the annulus $J \times \mathbb{I}$ in $\mathbb{S}^3 \times \mathbb{I}$. Then $J \asymp_f J$ via A_J and we have the covering projection $\Pi : (\Sigma \times \mathbb{I}) \times \mathbb{R} \rightarrow \overline{\mathbb{S}^3 \times \mathbb{I} \setminus V(A_J)}$. Let $D : \mathbb{B}^2 \rightarrow \overline{\mathbb{S}^3 \setminus V(J)}$ denote the immersion of the ribbon disc for K and by abuse of notation, the ribbon disc itself. Using saddle moves and deaths on D , we see that $L \asymp_{sf} J \sqcup \bigcirc$, where \bigcirc bounds a disc in \mathbb{S}^3 disjoint from J .

Now let $(\mathfrak{k}^{\Sigma \times \mathbb{R}}, \Pi_J, K^{N_J})$ be a virtual cover. To see that v is ribbon, note that D lifts to an immersed disc $\mathfrak{d} : \mathbb{B}^2 \rightarrow \Sigma \times \mathbb{R}$ with $\partial\mathfrak{d} = \mathfrak{k}$ (essentially by the chain rule and lifting criterion). Consider a ribbon singularity of D as a path in D connecting two points of K . This lifts to a path in \mathfrak{d} connecting two points of \mathfrak{k} . Since $\Pi_J(\mathfrak{d}) = D$, these are ribbon singularities of \mathfrak{d} . Moreover, \mathfrak{d} can have no other singularities. Using saddle moves and deaths on \mathfrak{d} in $(\Sigma \times \mathbb{I}) \times \mathbb{R}$, we see that \mathfrak{k} is ribbon. Hence v is a ribbon virtual knot. \square

Example: Recall that $J \sqcup K$ is a *ribbon link* if it is the boundary of an immersed $\mathbb{B}^2 \sqcup \mathbb{B}^2$ all of whose singularities are ribbon singularities. The left hand side of Figure 6 shows a two component link $L = J \sqcup K$ both of whose components are ribbon. Indeed, J is a square

knot and K is a trivial knot. The square knot is a fibered knot with a fiber Σ_J drawn as in the figure. Note that K is in SSF relative to Σ_J . An invariant associated virtual knot v can be found. Note that $w_v(t) = -2t \neq 0$. Hence, K cannot bound a ribbon disc in the complement of J . This also implies that L is not a *pure ribbon link* i.e. a ribbon link where the components bound disjoint ribbon discs.

A *boundary link* is a link where the components bound disjoint Seifert surfaces. Any pure ribbon link is a boundary link because the ribbon singularities may be modified as in [8] to obtain Seifert surfaces. Cochran's generalized Sato-Levine invariant $\beta(L)$ [7] is a \mathbb{Z}^∞ -valued link concordance invariant that is equal to $(0, 0, 0, \dots)$ on boundary links, hence on pure ribbon links. To compute the invariant begin with a link $J \sqcup K$ with $\text{lk}(J, K) = 0$ and find a Seifert surface Σ_K of K that does not intersect J . Σ_K may be chosen so that $\Sigma_K \cap \Sigma_J$ is connected; call this component a *derivative* $D(L)$. It is oriented following the convention of [7]. The first coordinate of the Sato-Levine invariant is $\text{lk}(D(L)^+, D(L))$, where $D(L)^+$ is the positive push off of $D(L)$ from Σ_J . This process is iterated to find the remaining coordinates.

For our link $L = J \sqcup K$, a convenient $D(L)$ is given on the right of Figure 6. The first coordinate of $\beta(L)$ is zero. It is easy to see that $J \sqcup D(L)$ is a ribbon link (e.g. it is a symmetric union [12]). All the remaining coordinates in $\beta(L)$ are zero. Thus the associated virtual knot detects that L is not a pure ribbon link, whereas $\beta(L)$ does not. Note that the Jones polynomial nullity detects this link as non-ribbon [11]. However the associated virtual knot detects something subtler: every ribbon disc for K must intersect J . \square

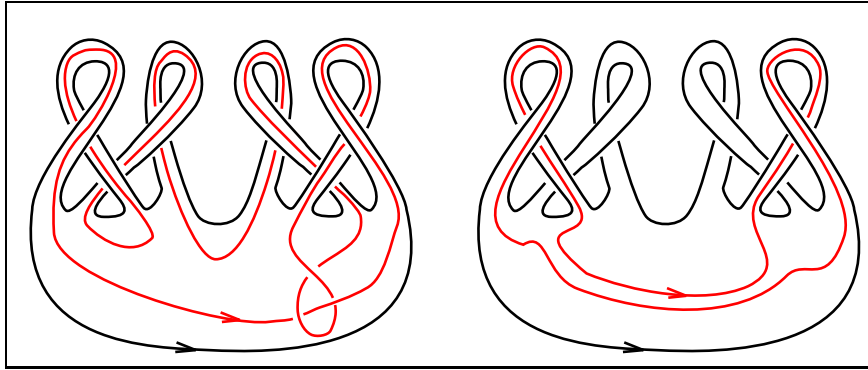


FIGURE 6. A link that is not pure ribbon (left) and a derivative used for computation of the Sato-Levine Invariant (right).

Example: We return now to our motivating examples from [14]. Recall they are an infinite family of two component slice links $L_n = J_0 \sqcup V_n$ with J_0 a square knot and V_n a smooth simple closed curve on a fiber Σ_J of J (by [26]). Thus, V_n is in SSF with respect to Σ_J . The SSF for V_n has no crossings in balls, so the invariant associated virtual knot $v_n \neq \bigcirc$. Since v_n is a ribbon virtual knot, one might hope to construct a ribbon presentation for L_n using virtual covers. Such a construction would require a fibered concordance of J_0 with the unknot obtained by pushing an immersed ribbon disc for J_0 into \mathbb{B}^4 . A method for doing exactly this was discovered by Aitchison and Silver [1].

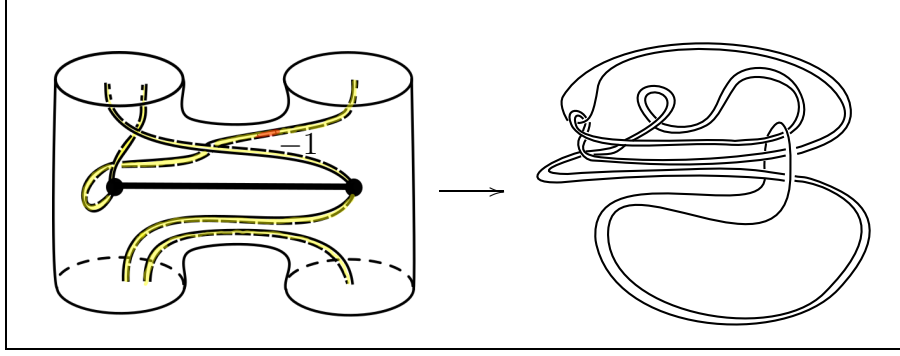


FIGURE 7. Construction of a ribbon disc for the square knot using the method of Aitchison and Silver.

Here we describe the method in brief. A slice disc D' of J_0 in \mathbb{B}^4 together with the fiber Σ_J bound a solid two holed torus H . The Alexander polynomial of J_0 specifies an automorphism ϕ of $\pi_1(H, *) \cong \langle x_1, x_2 | - \rangle$ (called the *monodromy*). An o.p. diffeomorphism $\psi : H \rightarrow H$ is constructed from ϕ . On the left in Figure 7, H is obtained by identifying each of the ellipses on top to the ellipse immediately below it. The yellow bands indicate the image of x_1 and x_2 under ϕ , as in [1]; the thick dumbbell represents possible positions for $*$. Lastly, a handlebody decomposition of $\overline{\mathbb{B}^4 \setminus V(D')}$ is recovered from the mapping torus defined by ψ . The ribbon disc D on the right in Figure 7 is also obtainable from the handlebody decomposition.

Problem: Either construct a ribbon presentation for each L_n such that the ribbon disc for J_0 is D (see Figure 7, right), or show that no such presentation exists. \square

4. INJECTIVITY OF SATELLITE OPERATORS

Take a knot P in a solid torus and tie it into the shape of a knot K in \mathbb{S}^3 . Roughly speaking, this is the satellite knot $P(K)$ with pattern P and companion K . For a fixed pattern P , do non-concordant companions yield non-concordant satellites with pattern P , i.e. is the map $K \rightarrow P(K)$ injective in concordance? Here we consider this question for the map $J \sqcup K \rightarrow J \sqcup P(K)$ in semi-fibered concordance. We begin with some preparatory results.

4.1. Satellite Operators and the HT polynomial. Let K be a knot in \mathbb{S}^3 and l be a longitude on a closed tubular neighborhood $\overline{V}(K)$ with $\text{lk}(K, l) = 0$. Let P be a knot in $\overline{V} = \mathbb{S}^1 \times \mathbb{B}^2$ that is not contained in a 3-ball in B . Let $f : \overline{V} \rightarrow \overline{V}(K)$ be an o.p. diffeomorphism such that $f(\mathbb{S}^1 \times \{0\})$ is identified with K , f takes a meridian of \overline{V} to a meridian of $\overline{V}(K)$, and the longitude $\mathbb{S}^1 \times \{1\}$ is mapped to l . Then the image $P(K)$ of P under f in \mathbb{S}^3 is an oriented knot called the *untwisted satellite* with *pattern* P and *companion* K . The knot $\mathbb{S}^1 \times \{0\}$ in \overline{V} represents a generator of $H_1(\overline{V}; \mathbb{Z}) \cong \mathbb{Z}$ and $[P]$ some integer q times this generator. Define the *absolute winding number* of P to be $r(P) = |q|$. The map $K \rightarrow P(K)$ is called a *satellite operator*.

A *classical satellite* (compare with [28]) of a virtual knot is defined as follows. Let v be a virtual knot diagram. Replace each arc of the diagram with p parallel arcs in \mathbb{R}^2 . At each

overcrossing/undercrossing arc of v , the p parallel arcs all pass over/under, respectively. Mark p^2 new virtual crossings for each virtual crossing of v . Lastly, break the p strands at some point away from the classical and virtual crossings, and insert an oriented (p, p) -tangle τ such that the result is an oriented knot. We further require that all the crossings in τ be classical. See Figure 8. The resulting virtual knot diagram $\tau(v)$ will be called a *classical satellite with companion v* . Let $r(\tau)$ denote the difference in the number of incoming and outgoing strands from the top of τ , in absolute value.

An untwisted satellite with companion K can clearly be represented as a classical satellite of the classical knot K . For links in SSF we have the following similar result.

Lemma 9. *Let $J \sqcup K$ be a two component link with J fibered and K in SSF with respect to some fiber Σ_J . Let P be a pattern. Then $J \sqcup P(K)$ is in SSF with respect to Σ_J and there is a (non-unique) (p, p) -tangle τ such that $r(P) = r(\tau)$ and $\kappa([P(K); \Sigma_J]) \rightleftharpoons \tau(\kappa([K; \Sigma_J]))$.*

Proof. Obtain a (p, p) -tangle τ' by cutting \bar{V} along a meridional disc intersecting P transversely. For each of the arcs in the SSF of K , both those lying in Σ_J and in the embedded 3-balls B_i , draw p parallel arcs. Let β be a (p, p) -tangle representing a sufficient number of full twists of the p strands to satisfy the requirement that $\text{lk}(K, l) = 0$, for a longitude l of K . Insert a diagram of $\tau = \tau' \cdot \beta$ on Σ_J into a portion away from the crossings of $[K; \Sigma_J]$. The resulting link is $J \sqcup P(K)$. By construction of τ , $P(K)$ is in SSF with respect to Σ_J , $r(P) = r(\tau')$, and $\kappa([P(K); \Sigma_J]) \rightleftharpoons \tau(\kappa([K; \Sigma_J]))$. \square

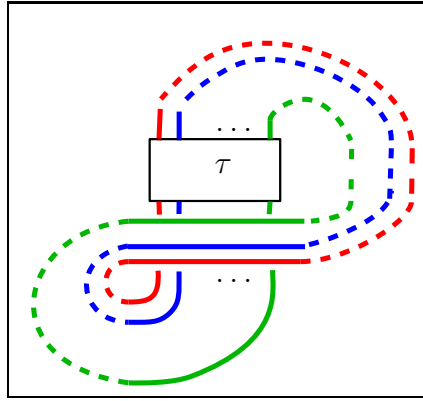


FIGURE 8. A schematic of a classical satellite $\tau(v)$ of a virtual knot v . The p strands are dyed different colors, but all lie one component.

The following theorem generalizes a result of A. Gibson [13] on cables to classical satellites. It is used in the next section to study injectivity of satellite operators in sf-concordance.

Theorem 10. *Let v be a virtual knot diagram and $\tau(v)$ a classical satellite with companion v . Let $r = r(\tau)$. Then $w_{\tau(v)}(t) = r^2 w_v(t^r)$.*

Proof. Consider each of the p parallel strands drawn in the construction of $\tau(v)$ to be dyed a different color, labeled as c_1, \dots, c_p . If x is a classical crossing of v , let $\tau(x)$ denote the set of p^2 corresponding crossings in $\tau(v)$. Thus, every classical crossing of $\tau(v)$ is either in $\tau(x)$ for some x or in τ . For each crossing y of $\tau(v)$, we will compute its index by traversing the

knot diagram from y and counting contributions of $\text{int}_y(a)$ until the first time we return to y .

Note that an entire dyed strand of color c_i is traversed before returning, then the net contribution of those crossings passed is zero. This is clear for crossings of the strands c_i with itself, since the trip passes both over and under the crossing. Now consider a crossing $z_1 \in \tau(x)$ where c_i crosses c_j and $j \neq i$. Observe that there is exactly one other crossing $z_2 \in \tau(x)$ such that c_i crosses c_j . It is easy to see that $\text{int}_y(z_1) = -\text{int}_y(z_2)$ regardless of how the strands happen to be oriented by τ . The net change to $\text{index}(y)$ is thus zero.

Consider first a classical crossing y in τ . Then the only contribution to its index comes from other crossing in τ . Thus we may replace all crossings in $\tau(v)$ outside of τ with virtual crossings without affecting $\text{index}(y)$. This results in a classical knot, hence $\text{index}(y) = 0$.

Now consider $y \in \tau(y_0)$ where y_0 is a classical crossing of v . For all $w \neq y_0$, convert all crossings in $\tau(w)$ to virtual ones. As detour moves do not affect the index, we may assume that τ and the crossings of $\tau(y_0)$ are positioned as in Figure 8. This virtual knot is classical, so the total contributions to $\text{index}(y)$ from crossings in τ and those in $\tau(y_0)$ is zero. Hence, we count only crossings in $\tau(w)$ with $w \neq y_0$. Each contribution to $\text{index}(y_0)$ in v counts $\pm r$ times in $\text{index}(y)$. Hence, $\text{index}(y) = \pm r \cdot \text{index}(y_0)$. Lastly note that the sum of signs of crossings in $\tau(y_0)$ is r^2 (resp. $-r^2$) when y_0 is signed \oplus (resp. \ominus). \square

4.2. Injectivity of Satellite Operators. A satellite operator P is said to be *injective* if $K_0 \not\asymp K_1$ implies $P(K_0) \not\asymp P(K_1)$ for all knots K_0, K_1 in \mathbb{S}^3 . In [9], the injectivity of P having strong winding number ± 1 was established for the topological, exotic, and smooth categories (assuming the smooth Poincaré Conjecture on \mathbb{S}^4).

Consider the satellite operator $J \sqcup K \rightarrow J \sqcup P(K)$. We will say that P is *injective in semi-fibered concordance* if $J_0 \sqcup K_0 \not\asymp_{sf} J_1 \sqcup K_1$ implies $J_0 \sqcup P(K_0) \not\asymp_{sf} J_1 \sqcup P(K_1)$.

Example: Embed the standard ribbon disc of the square knot in a solid torus so that it is not contained in a 3-ball. Let P be the pattern corresponding to this square knot. Let $J \sqcup K$ be the link on the left in Figure 6. By Theorem 8 (2), $J \sqcup P(K) \asymp_{sf} J \sqcup \bigcirc \asymp_{sf} J \sqcup P(\bigcirc)$, where \bigcirc bounds a disc in the complement of J . However, we showed that $J \sqcup K \not\asymp_{sf} J \sqcup \bigcirc$ since the associated virtual knot has non-vanishing HT polynomial. Thus P is not injective on the set of all semi-fibered concordance classes of two-component links. \square

Two links in SSF whose associated virtual knots have different HT polynomials are not semi-fibered concordant. This condition is sufficient to prove injectivity in semi-fibered concordance of non-zero absolute winding number satellite operators applied to such links.

Theorem 11. *For $i = 0, 1$, let $J_i \sqcup K_i$ be a two component link with J_i fibered, K_i in SSF with respect to some fiber Σ_i of J_i , and invariant associated virtual knot $v_i = \kappa([K_i; \Sigma_i])$. Let P be a pattern with $r(P) \neq 0$. If $w_{v_0}(t) \neq w_{v_1}(t)$, then $J_0 \sqcup P(K_0) \not\asymp_{sf} J_1 \sqcup P(K_1)$.*

Proof. Let τ' be a (p, p) -tangle obtained by cutting the solid torus containing P along a meridional disc. By Lemma 9 and its proof, for $i = 0, 1$ there is a (p, p) -tangle $\tau_i = \tau' \cdot \beta_i$ such that $\kappa([P(K_i); \Sigma_i]) \asymp \tau_i(v_i)$, where β_i is some appropriate number of full twists for K_i .

Suppose that $J \sqcup P(K_0) \asymp_{sf} J \sqcup P(K_1)$. By Theorems 6 and 1, $w_{\tau_0(v_0)}(t) = w_{\tau_1(v_1)}(t)$. Since $r(P) \neq 0$, Theorem 10 implies that $w_{v_0}(t^{r(P)}) = w_{v_1}(t^{r(P)})$. This is a contradiction. \square

5. CONCORDANCE AND CABLES OF KNOTS IN 3-MANIFOLDS

5.1. Concordance in 3-Manifolds. Virtual covers can also be used to study concordance of knots in closed oriented 3-manifolds N . In this section we sketch some applications and examples. We will say that oriented knots K_0^N and K_1^N are concordant in N if there is a properly embedded annulus A in $N \times \mathbb{I}$ such that for $i = 0, 1$, $A \cap (N \times \{i\}) = (-1)^i K_i$. As usual, we write $K_0^N \asymp K_1^N$ to denote concordance in N .

Theorem 12. *Let $(\mathfrak{k}_0^{\Sigma \times \mathbb{R}}, \Pi, K_0^N), (\mathfrak{k}_1^{\Sigma \times \mathbb{R}}, \Pi, K_1^N)$ be virtual covers with invariant associated virtual knots v_0, v_1 , respectively. If $K_0^N \asymp K_1^N$, then $v_0 \asymp v_1$.*

Proof. It is similar to the proof of Theorem 6, and hence we leave it as an exercise. \square

A 3-manifold N is said to be *fibred* if it can be represented as a mapping torus $\Sigma \times \mathbb{I}/\psi$, where $\psi : \Sigma \rightarrow \Sigma$ is an orientation preserving diffeomorphism. As such it is a fiber bundle over \mathbb{S}^1 with fiber Σ . We will assume Σ is c.c.o., so that the covering space $\Pi : \Sigma \times \mathbb{R} \rightarrow N$ defined by the mapping torus is regular and orientation preserving. Similar to knots in fibered knot complements, we can define *special surface form* (SSF, again), where a knot can be decomposed into arcs in a fixed fiber and “crossings” in small balls, each intersecting Σ is a disc.

When comparing knots in N , we will always state the explicit hypothesis that both knots are in SSF with respect to the *same fiber* of a given fibration/mapping torus. Under this condition, as in [6], it follows that a knot in SSF has an invariant associated virtual knot and equivalent knots in SSF with respect to the same fiber have equivalent invariant associated virtual knots.

Example: A handlebody decomposition (from Exercise 8.2.4 of [15]) of a 4-manifold whose boundary is a fibered 3-manifold N , is given in the center picture in Figures 9 and 10. Here n is any integer. A torus fiber can be seen as follows. Let D be the visible disc contained in the plane of the paper, bounded by the four black 0-framed arcs and arcs intersecting the boundaries of the attaching regions of the 1-handles. In the 4-manifold, this is a torus with a disc removed. The 0-framed arcs, viewed as a 3-manifold Dehn surgery, attach a disc along the boundary of the removed disc to create a fiber $\Sigma \approx \mathbb{S}^1 \times \mathbb{S}^1$. The $-1/n$ -framed arc in the diagram corresponds to the orientation preserving diffeomorphism in the mapping torus $\psi^n : \Sigma \rightarrow \Sigma$, where ψ is a Dehn twist (see [15]).

Let K_0^N be the red knot indicated in the middle of Figure 9. Let K_1^N be the red knot indicated in the middle of Figure 10. The left hand side of Figure 9 (resp., Figure 10) shows K_0^N (resp., K_1^N) in the more conventional form of a *mixed link diagram* [24]. Both K_0^N and K_1^N are in SSF with respect to Σ . The invariant associated virtual knots v_0, v_1 are given on the right hand side of their respective figures. Since $w_{v_0}(t) = 2t \neq 0 = w_{v_1}(t)$, it follows that $K_0^N \not\asymp K_1^N$. Note that these knots represent the same homology class when considered as curves on Σ . By Theorem 2, the slice genus of both v_0 and v_1 is 1, so neither K_0^N nor K_1^N is concordant to the boundary of an embedded disk in N . \square

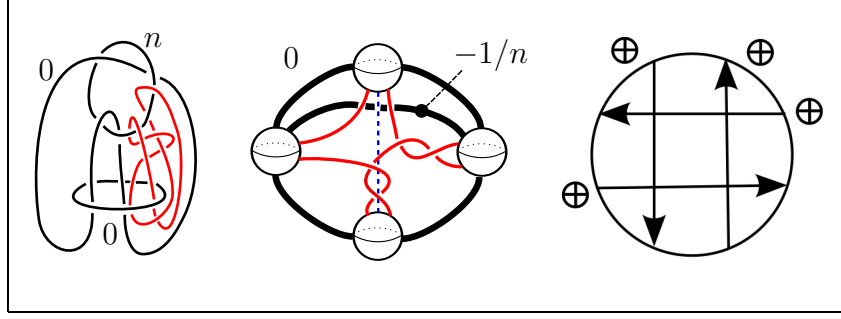


FIGURE 9. A knot K_0^N in a fibered 3-manifold N (left). A Gauss diagram of the invariant associated virtual knot v_0 to K_0^N (right).

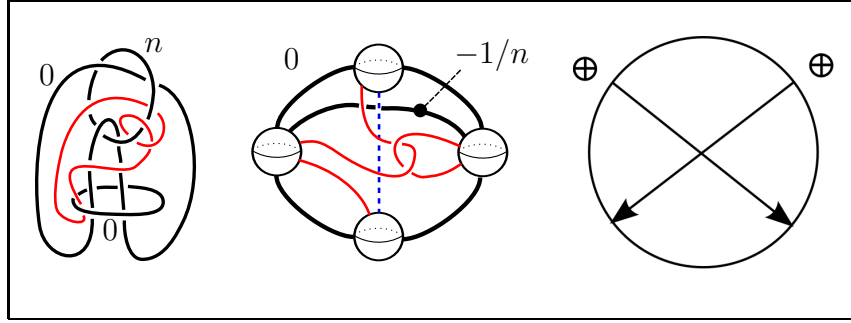


FIGURE 10. A knot K_1^N in a fibered 3-manifold N (left). A Gauss diagram of the invariant associated virtual knot v_1 to K_1^N (right).

5.2. Cables and Irreducible 3-Manifolds. A 3-manifold is said to be *irreducible* if every embedded \mathbb{S}^2 bounds an embedded \mathbb{B}^3 . Otherwise, it is said to be *reducible*. A link L in \mathbb{S}^3 is split if and only if $\overline{\mathbb{S}^3 \setminus V(L)}$ is reducible. If $L = J \sqcup K$ is a two component split link with J fibered and K in SSF with respect to a fiber Σ_J , then K is the invariant associated virtual knot [4]. Thus $\overline{\mathbb{S}^3 \setminus V(L)}$ is irreducible whenever the invariant associated virtual knot is non-classical. Here we apply this idea to Dehn surgeries on knots in fibered 3-manifolds that yield reducible 3-manifolds.

Let N be a closed oriented 3-manifold and K^N a knot with fixed trivialization of the normal bundle of K . Let P be a knot in $\overline{V} = \mathbb{B}^2 \times \mathbb{S}^1$ not contained in an embedded 3-ball in \overline{V} . As in the case of $N = \mathbb{S}^3$, we may define a satellite operator via a longitude l of a closed tubular neighborhood $\overline{V}(K)$ and an o.p. diffeomorphism $f : \overline{V} \rightarrow \overline{V}(K)$ identifying $f(\mathbb{S}^1 \times \{0\})$ with K , a meridian of \overline{V} to a meridian of $\overline{V}(K)$, and the longitude $\mathbb{S}^1 \times \{1\}$ to l . Set $P(K, l)^N := f(P)^N$.

Let $S_{1/2}$ be the circle of radius $1/2$ centered at 0 in \mathbb{B}^2 . If P as above is equivalent in \overline{V} to a non-trivial (p, q) torus knot on the torus $S_{1/2} \times \mathbb{S}^1$, then $P(K, l)^N$ is said to be *cabled*.

Theorem 13. *Let K^N be a knot in a fibered 3-manifold in SSF with respect to a fiber Σ and v the invariant associated virtual knot. If v is non-classical and there is a non-trivial Dehn surgery on K that yields a reducible 3-manifold, then K is cabled in N .*

Proof. Let $p : N \rightarrow \mathbb{S}^1$ denote the fiber bundle projection. Scharlemann and Thompson proved [27] that the reducibility condition implies either: (1) $p(K)$ has non-zero degree, (2) K is contained in an embedded 3-ball, (3) K is cabled in N , or (4) K is a simple closed curve in a fiber. Since K lifts to a knot in $\Sigma \times \mathbb{R}$, option (1) is impossible. If K is contained in an embedded 3-ball, then v must be classical. If K is a simple closed curve in a fiber, its invariant associated virtual knot is equivalent to the trivial knot. Thus the hypotheses on v eliminate all options but (3). \square

We remark that the invariant associated virtual knot can be used to eliminate some types of cabling of K^N . Suppose for example that K^N is cabled so that the companion C^N is in SSF with respect to a fiber. Then there is a virtual cover $(\mathfrak{k}^{\Sigma \times \mathbb{R}}, \Pi, K^N)$ such that $\mathfrak{k}^{\Sigma \times \mathbb{R}}$ is cabled with companion $\mathfrak{c}^{\Sigma \times \mathbb{R}}$, where $(\mathfrak{c}^{\Sigma \times \mathbb{R}}, \Pi, C^N)$ is a virtual cover. By Theorem 10, the HT polynomial of the invariant associated virtual knot v of K^N depends only on $\kappa(\mathfrak{c}^{\Sigma \times \mathbb{R}})$ and $r(P)$, where P is the torus knot pattern of K^N . Thus $w_v(t)$ can exclude such C^N .

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